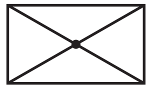
ROTATIONAL MOTION 01

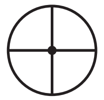
PHYSICS WALLAH

Centre of Mass

- Avg. position of all the parts of the system, weighted according to their mass
- For homogeneous object, centre of mass lie at their geometric centres



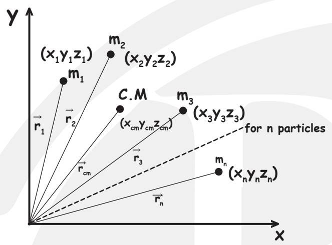
- Centre of mass, may or may not lies inside the object



- Centre of mass, may change it's location

Centre of Mass For System of n Particles

$\vec{r}_{cm} = \dfrac{\Sigma m\vec{r}}{\Sigma m}$



General Equation

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \text{-----} m_n\vec{r}_n}{m_1 + m_2 + m_3 + \text{------} + m_n}$$

In terms of Cartesian co-ordinates

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \text{-----} m_nx_n}{m_1 + m_2 + m_3 + \text{------} + m_n}$$

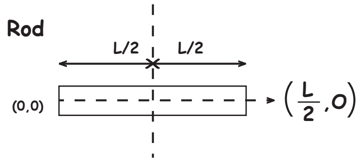
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \text{-----} m_ny_n}{m_1 + m_2 + m_3 + \text{------} + m_n}$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \text{-----} m_nz_n}{m_1 + m_2 + m_3 + \text{------} + m_n}$$

Centre of mass for various shapes

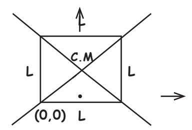
Uniformly distributed mass — centre of mass

1) Rod



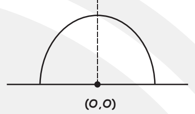
$\left(\frac{L}{2}, 0\right)$

2) Square Lamina



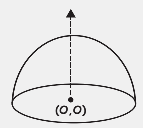
$\left(\frac{L}{2}, \frac{L}{2}\right)$

3) Semicircular Ring



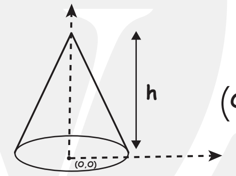
$\left(0, \frac{2R}{\pi}\right)$

4) Hemispherical shell



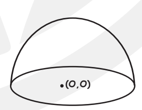
$\left(0, \frac{R}{2}\right)$

5) Solid circular cone



$\left(0, \frac{h}{4}\right)$

6) Solid hemispherical



$\left(0, \frac{3R}{8}\right)$

Motion centre of mass

velocity of centre of mass

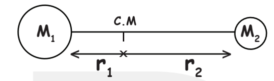
$$\vec{V}_{cm} = \frac{M_1\vec{V}_1 + M_2\vec{V}_2 + M_3\vec{V}_3 + \text{--}}{M_1 + M_2 + M_3 + \text{------}}$$

Acceleration of centre of mass

$$\vec{a}_{cm} = \frac{M_1\vec{a}_1 + M_2\vec{a}_2 + M_3\vec{a}_3 + \text{--}}{M_1 + M_2 + M_3 + M_4 \text{------}}$$

Isolated System

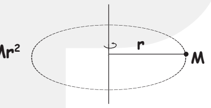
- No external forces acting on the system
- They can have mutual force of attraction



$M_1r_1 = M_2r_2 \quad a_{cm} = 0, v_{cm} = \text{constant}$

Moment of Inertia

$I = Mr^2$

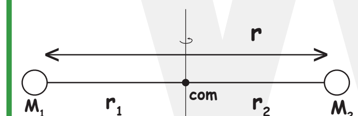


m= Mass of body
r= Perpendicular distance from the axis of rotation

Moment of Inertia
- Tensor Quantity
- $I = Mr^2$
- Rotational analogous of mass

Two Point Mass

$I_{COM} = M_1r_1^2 + M_2r_2^2$



$r_1 = \dfrac{m_2 r}{m_1 + m_2}$, $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

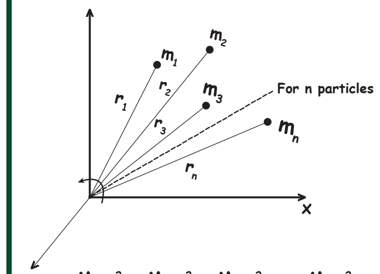
$I_{com} = m_{red}r^2$, $m_{red} = \dfrac{m_1m_2}{m_1 + m_2}$

Factors Affecting Moment of Inertia

- Mass of the body
- Axis of rotation
- Mass distribution

Moment of Inertia

i) For discrete system of particles

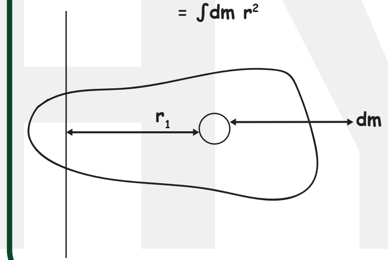


$$I = \frac{M_1r_1^2 + M_2r_2^2 + M_3r_3^2 + \text{---} M_nr_n^2}{M_1 + M_2 + M_3 + \text{------} M_n}$$

r_i = ⊥distance from axis of rotation

ii) For non-point mass

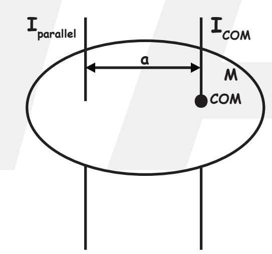
$I = \int dI = \int r^2dm$
$= \int dm\ r^2$



Parallel Axis Theorm

Conditions:-

1) Axis where considered to be parallel to each other
2) One of the axis must pass through centre of mass

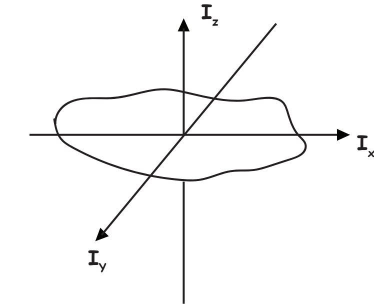


$I_{parallel} = I_{COM} + Ma^2$

Theorm of Perpendicular axis

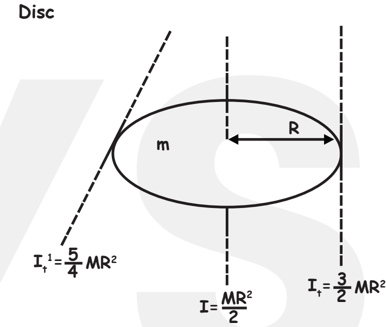
$I_z = I_x + I_y$ (Only valid for laminar bodies)

Note:-
X and Y axis lie in the plane of body
Z- axis ⊥ need to the plane of the body
Axis need not pass through center of mass

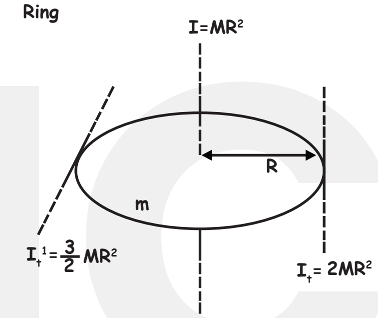


Moment Of Inertia For Various Objects

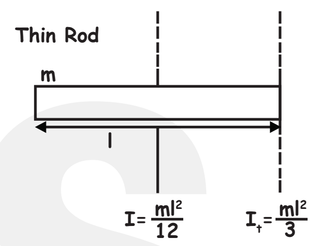
Disc



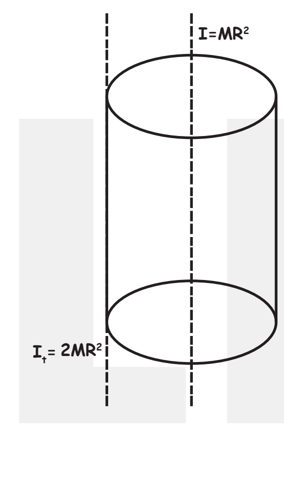
Ring



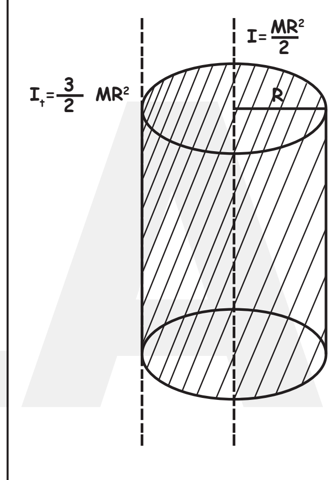
Thin Rod



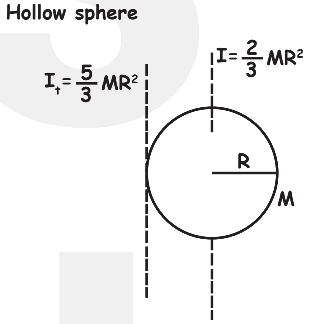
Hollow cylinder



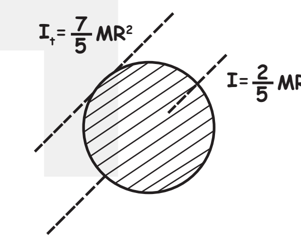
Solid cylinder



Hollow sphere



Solid sphere

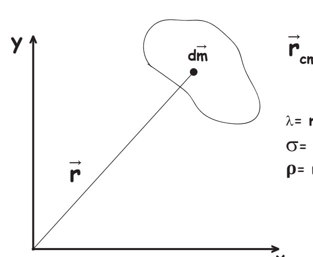


I=Moment of Inertia along the center of mass

I_t=Moment of Inertia along the tangent perpendicular to the plane

I_t'=Moment of Inertia along the tangent parallel to the plane

Centre Of Mass For Non Point Mass



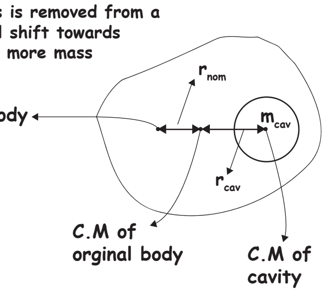
$\vec{r}_{cm} = \dfrac{\int dm.\vec{r}}{\int dm}$

λ= mass per unit length
σ= mass per unit area
ρ= mass per unit volume

1) Mass distributed over length ⇒ $dm = \lambda\, dl$
2) Mass distributed over area ⇒ $dm = \sigma\, dA$
3) Mass distributed over volume ⇒ $dm = \rho dV$

Cavity in object

If some mass is removed from a body C.M will shift towards the side with more mass

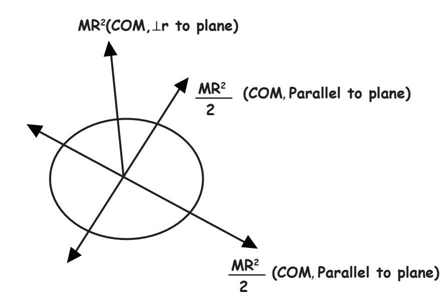


Assuming $r_{C.M}$ is at the origin

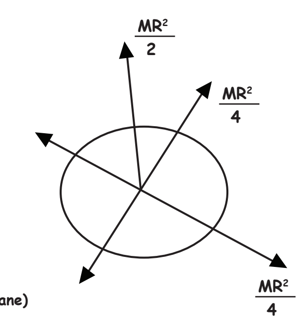
$$r_{rem} = \frac{-M_{cav} \times r_{cav}}{M_{rem}}$$

Moment of Inertia along the centre of Mass Perpendicular to the Plane Surface

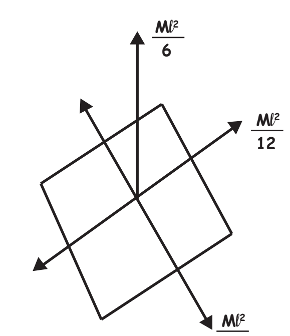
1) Ring



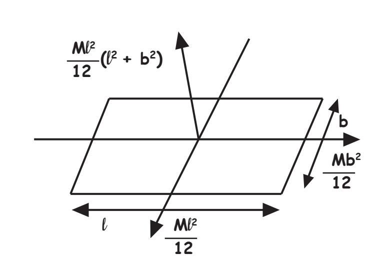
2) Disc



3) Square sheet



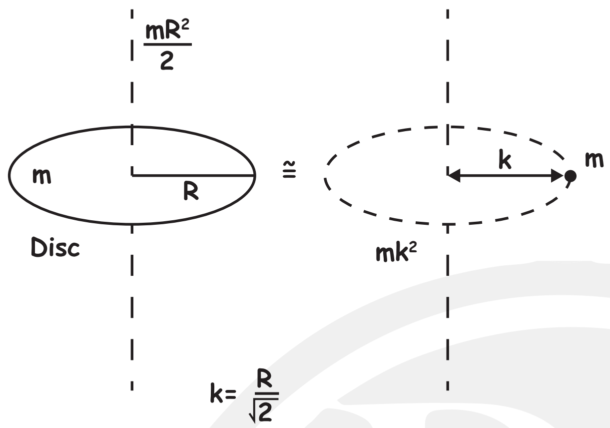
4) Rectangular sheet



ROTATIONAL MOTION 02

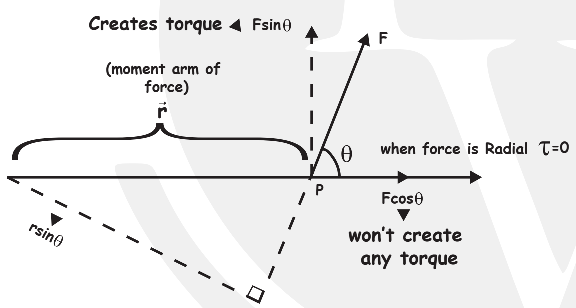
Radius of Gyration

Definition: The distance from the axis of a point mass whose mass is equal to the mass of whole body and whose moment of inertia is equal to moment of inertia of the body about the axis



k is the radius of gyration

Torque



Torque $\tau = r\,F\sin\theta$

$= F\sin\theta \times r = F_{\perp} r$ (1)

$= F \times r\sin\theta = F\, r_{\perp}$ (2)

$\vec{\tau} = \vec{r} \times \vec{F}$ (Vector form)

If force is radial ie $\theta = 0°$ or $180°$

Torque $\tau = 0$

If force is tangential and $\perp^r$ to radius vector $\theta = 90°$

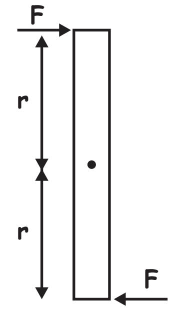
Torque, $\tau = \tau_{max} = rF$

Equilibrium

For translational equilibrium

$F_{net} = 0$

τ_{net} may or maynot be zero



Rotational Equillibrium

$\tau_{net} = 0$

F_{net} may or maynot be zero

Also called principle of moments

Static Equilibrium

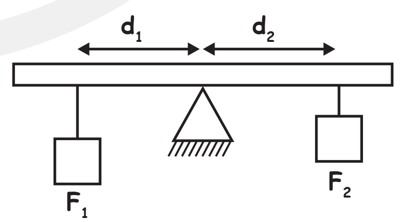
Combination of both translational and rotational equilibrium

$F_{net} = 0 \Rightarrow$ Forces are balanced

$\tau_{net} = 0 \Rightarrow \tau_{clockwise} = \tau_{anticlockwise}$

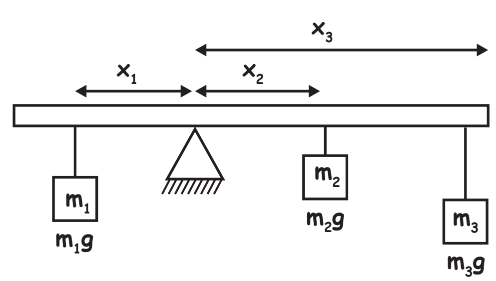
Principle of moments

When a body is in rotational equilibrium sum of clock wise moments about any point is equal to sum of anticlockwise moments about that point

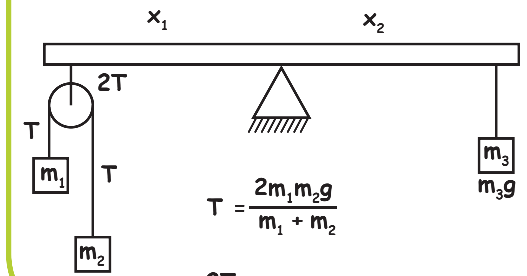


$F_1 \times d_1 = F_2 \times d_2$

Load × load arm = Effort × effort arm



$m_1gx_1 = m_2gx_2 + m_3gx_3$



$T = \frac{2m_1m_2g}{m_1 + m_2}$

$2Tx_1 = m_3x_2g$

Angular acceleration

$\tau = I\alpha$

τ - torque

I moment of inertia

α angular acceleration

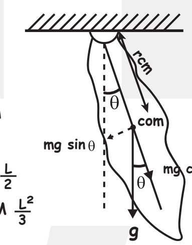
Initial angular acceleration when a rod is released

Initial angular acceleration when abody is released from an angle θ

$\alpha = \frac{(mg\sin\theta)r_{cm}}{I}$

For rod $r_{cm} = \frac{L}{2}$

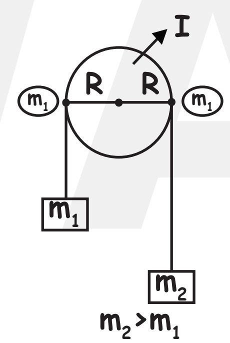
$I = M\frac{L^2}{3}$



Translatory - rotation combination

$\alpha = \frac{\tau_{applied} - \tau_{opposition}}{\text{Total } I}$

$\alpha = \frac{m_2Rg - m_1Rg}{m_1R^2 + m_2R^2 + I}$



$m_2 > m_1$

Angular momentum & its conservation

Angular momentum of a point mass:-

Angular momentum about origin

$\vec{L} = \vec{r} \times \vec{P}$

$= m(\vec{r} \times \vec{v})$

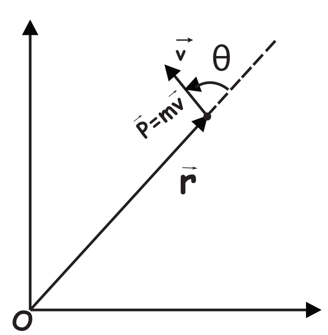
$\vec{P} = m\vec{v}$

1) When $\theta = 0°$ or $180°$

$L_o = mvr\sin 180 = 0$

$mvr\sin 0 = 0$

Angular momentum is minimum



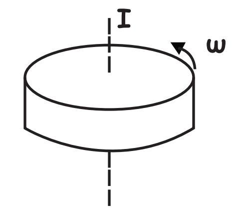
1) When $\theta = 90°$

$|\vec{L}| = r\,p\sin\theta$

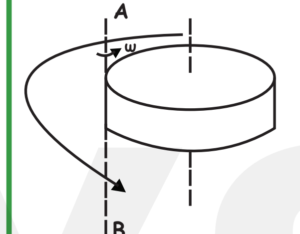
$= rp$

$= L_{max}$

Spin angular momentum



$L_{spin} = I\omega$



$L_{spin} = I_{AB}\omega$

Conservation of Angular momentum

If there is no external torque Angular momentum is conserved

- $\tau = \frac{dL}{dt}$
- If $\tau = 0 \Rightarrow \frac{dL}{dt} = 0$

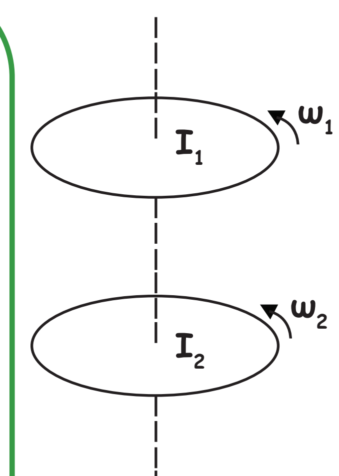
  L=constant

  $I\omega$=constant
- $I_1\omega_1 = I_2\omega_2$
- If moment of intertia increases angular velocity decreases and if moment of intertia decreases angular velocity increases

Moment of intertia when two discs joined:

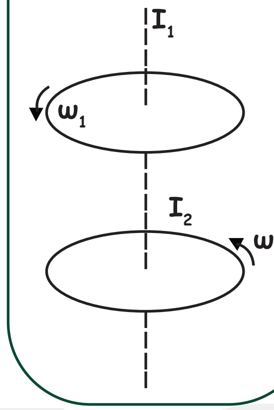
same direction:-

$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$



Opposite direction:-

$\omega_f = \frac{I_1\omega_1 - I_2\omega_2}{I_1 + I_2}$



Work,Energy & Power

1) Work done by a torque, W

$= \tau\theta$ (if torque is uniform)

$= \int \tau\, d\theta$ (if force is non uniform)

2) $K.E = \frac{1}{2}I\omega^2$ (when ω is constant)

$= \frac{L^2}{2I}$ (when L is constant)

$= \frac{1}{2}L\omega$

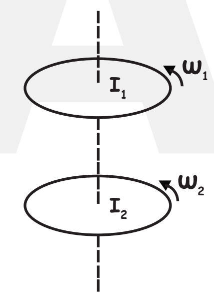
3) Work-Energy theorm

$\Sigma W = \Delta K = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$

Energy loss when 2 discs are joined:-

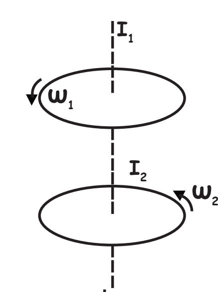
1) same direction:-

$\Delta K.E = \frac{I_1I_2}{2[I_1 + I_2]}(\omega_1 - \omega_2)^2$



2) Opposite direction:-

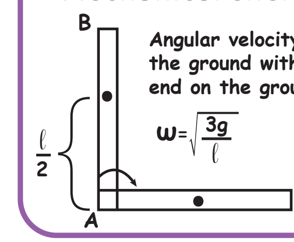
$\Delta K.E = \frac{I_1I_2}{2[I_1 + I_2]}(\omega_1 + \omega_2)^2$



Mechanical energy conservation

Angular velocity with which the rod hits the ground without slipping and one of its end on the ground, released from rest

$\omega = \sqrt{\frac{3g}{\ell}}$



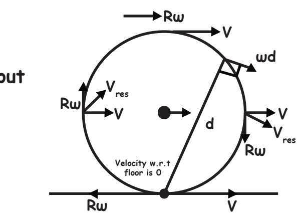
Rolling Motion

Translatory+Rotatory=Rolling

Velocity in rolling

- Condition for rolling without slipping:- $V = R\omega$
- Velocity of any point on rolling object, V_P

  $= \omega d = \frac{v\,d}{R}$



d is the distance from point content

Energy in rolling motion

1) Translatory Motion

$T_{K.E} = \frac{1}{2}mv^2$

2) Spinning motion/rotational motion

$R_{K.E} = \frac{1}{2}I\omega^2 = \frac{1}{2}mk^2\frac{V^2}{R^2} = \frac{1}{2}mv^2 \times \left(\frac{k^2}{R^2}\right)$

3) Rolling motion

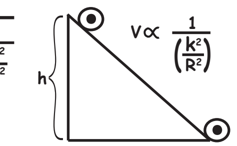
$T_{K.E} + R_{K.E} = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 \times \frac{k^2}{R^2} = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$

$\frac{k_{total}}{k_{trans}} = \left(1 + \frac{k^2}{R^2}\right)$

Motion on an inclined plane

Velocity at bottom $= \sqrt{\frac{2gh}{1 + \frac{k^2}{R^2}}}$

$V \propto \frac{1}{\sqrt{\left(\frac{k^2}{R^2}\right)}}$



$\frac{k^2}{R^2}\uparrow \Rightarrow V\downarrow \Rightarrow$ Time $\uparrow$

Velocity:solid sphere>Disc>Hollow>Sphere>Ring

Time to reach bottom:Ring>Hollow sphere >Disc>solid sphere

Value of velocity:-

1) Ring/Hollow cylinder $= \sqrt{gh}$

2) Disc/Solid cylinder $= \sqrt{\frac{4}{3}gh}$

3) Hollow sphere $= \sqrt{\frac{6}{5}gh}$

4) Solid sphere $= \sqrt{\frac{10}{7}gh}$

Acceleration

$a = \frac{g\sin\theta}{1 + \frac{k^2}{R^2}}$ $\quad a \propto \frac{1}{1 + \frac{k^2}{R^2}}$

Time of descend:-

$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1 + \frac{k^2}{R^2}\right)}$

$t \propto \sqrt{1 + \frac{k^2}{R^2}}$

Ring>Hollow sphere>Disc>solid sphere

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